The Canonical Form of the Rabi Hamiltonian

M. Szopa,[*] G. Mys and A. Ceulemans

Department of Quantum Chemistry, University of Leuven,

Celestijnenlaan 200F, B-3001 Leuven, Belgium.

Abstract

The Rabi Hamiltonian, describing the coupling of a two-level system to a single quantized boson mode, is studied in the Bargmann-Fock representation. The corresponding system of differential equations is transformed into a canonical form in which all regular singularities between zero and infinity have been removed. The canonical or Birkhoff-transformed equations give rise to a two-dimensional eigenvalue problem, involving the energy and a transformational parameter which affects the coupling strength. The known isolated exact solutions of the Rabi Hamiltonian are found to correspond to the uncoupled form of the canonical system.

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I. Introduction

In the study of dynamical problems the harmonic oscillator occupies a prominent place, as a prototype of the fundamental unitary symmetry group. The spell of group theory also extends to anharmonic oscillators, which have recently been exposed as mere q-deformations of the unitary group algebra [1]. In contrast dynamical problems which arise in the study of boson-fermion

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[*]On leave of the Institute of Physics, University of Silesia, Uniwersytecka 4, 40-007 Katowice, Poland

interactions are more reluctant to reveal their hidden symmetries. Such problems are more involved since the actual states of the system are of composite nature with a boson and a fermion part. This results in the appearance of singularities in the corresponding eigenvalue problem. Examples include the Jahn-Teller Hamiltonian in molecular physics, and the Rabi Hamiltonian in nuclear physics and quantum optics.

A rather peculiar feature of these systems is the emergence of isolated exact solutions, with eigenvalues that correspond to simple expressions in rational numbers. Such were obtained - in a rather heuristic way - by Judd [2] for the case of the $E \otimes e$ Jahn-Teller Hamiltonian, and by Kuś [3] for the spectrum of a two-level atom coupled to a single quantized mode. It was suggested that the exact solutions probably hint at some dynamical symmetry group - but so far no progress was reported in this direction.

In the present paper we analyze the one mode Rabi problem in a more rigorous way. First - following [3] and [4] - the dynamical problem is defined in the Bargmann-Fock Hilbert space of entire functions. The resulting system of differential equations is then put into canonical form using a theorem due to Birkhoff. Under this transformation the Kuś exact solutions are found to be mapped onto the levels of a displaced harmonic oscillator. In this way hidden symmetry appears.

II. The Bargmann-Fock Representation of a Hilbert Space

Before turning to the actual results we briefly review basic information about the Hilbert space of entire functions introduced by Fock [5] and Bargmann [6]. Let $\mathcal{F}_n$, for n integral, be the set of entire analytic functions $f(z)$, where $z = (z_1, ..., z_n)$ and $z_k \in \mathbf{C}$ are complex numbers, $k = 1, ..., n$. Because $f(z)$ is entire it has an everywhere converging power series

$$f(z) = \sum_{k_1, ..., k_n} \alpha_{k_1, ..., k_n} z_1^{k_1} \cdot ... \cdot z_n^{k_n}, \tag{1}$$

where summation extends over the whole set of non-negative integers $k_1, ..., k_n$ and $\alpha_{k_1,...,k_n}$ are complex coefficients. We define an inner product of two elements f and g of $\mathcal{F}_n$ by

$$(f, g) = \int \overline{f(z)} \cdot g(z) d\mu_n(z), \tag{2}$$

where $\overline{f(z)} \cdot g(z)$ is a usual scalar product in $\mathbf{C}^n$,

$$d\mu_n(z) = \frac{1}{\pi^n} \exp(-\overline{z} \cdot z) \prod_{k=1}^{n} dx_k dy_k, \qquad z_k = x_k + iy_k \tag{3}$$

and the integration extends over the whole space $\mathbf{C}^n$. The Bargmann-Fock space $\mathcal{F}_n$ is a set of all entire functions (1), which have a finite norm $(f, f) < \infty$. This is equivalent to the requirement that

$$|f(z)| \leq c \exp(\frac{1}{2}\gamma \overline{z} \cdot z), \tag{4}$$

where c and γ are positive constants with $\gamma < 1$. The Bargmann-Fock space $\mathcal{F}_n$ with the inner product defined by (2) is a Hilbert space.

Let us now consider two operators in $\mathcal{F}_n$: multiplication by z_k and differentiation $\frac{d}{dz_k}$. Since the functions $f(z)$ of $\mathcal{F}_n$ are analytic, $z_k f$ and $\frac{d}{dz_k} f$ always exist. The operators satisfy the commutation rules

$$[z_k, z_l] = 0, \qquad \left[\frac{d}{dz_k}, \frac{d}{dz_l}\right] = 0, \qquad \left[\frac{d}{dz_k}, z_l\right] = \delta_{kl}. \tag{5}$$

Furthermore, with respect to the inner product (2) z_k and $\frac{d}{dz_k}$ are Hermitian conjugate

$$(z_k f, g) = \left(f, \frac{d}{dz_k} g\right) \tag{6}$$

whenever $z_k f$ and $\frac{d}{dz_k} f$ belong to $\mathcal{F}_n$.

On the other hand relations (5) and (6) are well known algebraic relations defining Hermitian conjugate annihilation a_k and creation a_k^+ operators of boson fields in second quantization,

$$\left[a_k^+, a_l^+\right] = 0, \qquad [a_k, a_l] = 0, \qquad \left[a_k, a_l^+\right] = \delta_{kl}. \tag{7}$$

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We conclude therefore that within $\mathcal{F}_n$ the annihilation operator a_k is represented by the operation $\frac{d}{dz_k}$, and the creation operator a_k^+ corresponds to the multiplication by z_k.

As an instructive example let us take a set of n identical uncoupled harmonic oscillators which are described up to a constant by the Hamiltonian $H = \sum_{l=1}^{n} a_l^+ a_l$. The corresponding operator in the Bargmann-Fock space (denoted by $\mathcal{H}$) is $\mathcal{H} = \sum_{l=1}^{n} z_l \frac{d}{dz_l}$ and the corresponding eigenproblem is the system of differential equations

$$\sum_{l=1}^{n} z_l \frac{d}{dz_l} f_k(z) = E_k f_k(z). \tag{8}$$

It is easily verified that the eigenfuctions in this case are functions of the type (1), which are homogeneous polynomials of the order k i.e. in (1) the sum is over $k_1 + ... + k_n = k$ and the corresponding eigenvalues are $E_k = k$.

III. The Transformation to a Canonical Form

In the Bargmann-Fock representation of a Hilbert space quantum mechanical equations for bosons interacting with a manifold of fermion states are represented by a system of linear differential equations in the complex domain [3, 4]. The physical solutions (1) of such a system must belong to $\mathcal{F}_n$ i.e. be entire and obey condition (4). In practice the solution of this equation is complicated due to the occurrence of finite regular singularities.

In this section we describe a transformation - due to Birkhoff [7] - which allows, for a system of linear equations of the first order in one variable, to find a canonical form. To be in the canonical form the system must be transformed in such a way that all its finite singularities reduce to only one singularity at zero, while at the same time preserving the order of the singularity at infinity. Consequently, the transformed system is more likely to be exactly solvable.

The system of m linear differential equations of the first order has a

general form

$$\frac{df_r}{dz} = \sum_{s=1}^{m} p_{rs}(z)f_s, \qquad\qquad r = 1,...,m \qquad\qquad (9)$$

and we assume that $p_{rs}(z)$ are analytic functions of a complex variable apart from a finite number of regular singularities (even at infinity). Practically it means, that outside the circle $|z| = R$, which includes all the finite singular points, the coefficients may be expanded in a Laurent series

$$p_{rs}(z) = \sum_{k=-\infty}^{q} p_{rs}^{(k)} z^k, \qquad\qquad p_{rs}^{(k)} \in \mathbf{C}, \qquad\qquad (10)$$

where $q \geq -1$ and $q + 1$ is termed the *rank* of the singular point at infinity. In general the system (9) can have m independent sets of solutions $f_s^{(t)}(z)$, where $s = 1,...,m$ denotes different solutions within a set $t = 1,...,m$. Now we assume a linear transformation of the form

$$f_r(z) = \sum_{s=1}^{m} a_{rs}(z)F_s(z) \qquad\qquad (11)$$

where the coefficients $a_{rs}(z)$ are analytic at infinity and reduce at infinity to a unit matrix

$$a_{rs}(z) = \sum_{k=0}^{\infty} \frac{a_{rs}^{(k)}}{z^k}, \qquad\qquad a_{rs}^{(k)} \in \mathbf{C}, \quad a_{rs}^{(0)} = \delta_{rs}. \qquad (12)$$

In a sense this $\{a_{rs}(z)\}$ matrix could be said to contain all the finite singularities of the initial system. Under this transformation the original system (9) turns into a system of a slightly different form

$$z\frac{dF_r(z)}{dz} = \sum_{s=1}^{m} P_{rs}(z)F_s(z), \qquad\qquad r = 1,...,m. \qquad (13)$$

The coefficients of the transformed system are given by the equation

$$\{P_{rs}(z)\} = z\left(\left\{a_{rk}^{-1}(z)\right\}\{p_{kl}(z)\}\{a_{ls}(z)\} - \left\{a_{rk}^{-1}(z)\right\}\left\{\frac{d}{dz}a_{ks}(z)\right\}\right), \quad (14)$$

where $\{a_{rk}^{-1}(z)\}$ is the matrix inverse to the matrix $\{a_{rk}(z)\}$ and $\{a_{rk}(z)\}\{p_{ks}(z)\}$ denotes the matrix multiplication. Now we are in the position to formulate the Birkhoff theorem - the crucial result in our analysis.

The Birkhoff theorem states that *for every system of the type (9) there exists a transformation matrix (12) such that the coefficients $P_{rs}(z)$ of the transformed system (13) are polynomials of a degree not exceeding $q + 1$* [7].

There are several properties of the above transformation which should be stated here.

(i) All finite singular points of the initial system (9) coalesce to only one singularity at zero (because $P_{rs}(z)$ are polynomials). This is the most significant property of the transformation and because of it the system (13) may be termed the *canonical* form of the system (9).

(ii) The ranks of the singular points at infinity for both systems (9) and (13) are equal.

(iii) If a given set of solutions $f_s^{(t)}(z)$, $s = 1, ..., m$, belongs to the Bargmann-Fock space $\mathcal{F}_1$, then the corresponding transformed solutions $F_s^{(t)}(z)$, which are found due to the Birkhoff theorem, also belong to this space. This can easily by shown by adapting the treatment by Birkhoff to the case of entire functions.

The final property allows to reject all the solutions of the transformed system which do not belong to the Bargmann-Fock space as being non physical. However the inverse of this property is not automatically true, but requires a proper choice of the transformation matrix. We will come back to this point when discussing the actual solutions of the system under investigation.

A useful test to check if the solutions of the transformed system can be entire is given by the indicial equation of the transformed system [8]

$$det\{c_{rs} - \rho\delta_{rs}\} = 0 \qquad (15)$$

where $c_{rs} = P_{rs}(0)$, $r, s = 1, ..., m$. The solutions of (13) depend on the roots $\rho_1, ..., \rho_m$ of the indicial equation. There are several rules connected with this equation which indicate the possibility of existence of entire solutions and their degeneracy.

D_0. If none of the roots is a non-negative integer, the equation (13) has no entire solutions (because they are not analytic in the origin).

D_1. If one of the roots ρ_t is a non-negative integer and the remaining roots

are either non-integers or are equal to ρ_t, then there exists exactly one, up to linear dependence, set of analytic solutions of (13) and it is of the form

$$F_s^{(t)}(z) = z^{\rho_t} u_s(z), \qquad s = 1, ..., m \qquad (16)$$

where the $u_s(z)$ are analytic and $u_s(0) \neq 0$. Functions (16) are entire provided their radii of convergence are infinite.

D_2. In the remaining cases if two or more roots of the indicial equation are integers (at least one of them is non-negative), there exists at least one analytic set of solutions of the form (16), where ρ_t is the maximal integral root of (15). The remaining solutions corresponding to other integral roots are usually singular in the origin, but in exceptional cases may also be analytic.

It should be pointed out that the Birkhoff theorem is an existence theorem, which as such does not provide the actual form of the canonical equation nor the transformation matrix. In practice at least the canonical equation can usually be found relatively easy by the use of (14), keeping in mind that the $P_{rs}(z)$ are polynomials of a given degree. To this aim we invert the equation (14) and express it in terms of the relevant expansion coefficients. This yields, for every integer $l \geq 0$, a system of m^2 equations

$$\sum_{i=0}^{l} \left(\left\{ a_{rk}^{(l-i)} \right\} \left\{ P_{ks}^{(q+1-i)} \right\} - \left\{ p_{rk}^{(q-i)} \right\} \left\{ a_{ks}^{(l-i)} \right\} \right) = (l - q - 1) \left\{ a_{rs}^{(l-q-1)} \right\}, \quad (17)$$

where expansion coefficients for $\{p_{rk}(z)\}$, $\{a_{rk}(z)\}$ and $\{P_{rs}(z)\}$ are defined by (10), (12) and $P_{rs}(z) = \sum_{k=0}^{q+1} P_{rs}^{(k)} z^k$ respectively. We assume also that $\{a_{rk}^{(i)}\}$ and $\{P_{rk}^{(i)}\}$ with negative indices i are zero matrices.

This formula can now be used to find the coefficients $P_{rs}(z)$ of the transformed system. In this case we only need the equations corresponding to $l = 0, ..., q + 1$. Here the trivial case with $l = 0$ immediately yields:

$$\{P_{rs}^{(q+1)}\} = \{p_{rs}^{(q)}\}. \qquad (18)$$

For higher l, $1 \leq l \leq q + 1$ the resulting expansion coefficients in the transformed system may also depend on the $a_{ks}^{(1)}, ..., a_{ks}^{(q+1)}$ coefficients in the expansion of the transformation matrix. These coefficients thus may enter the

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transformed system as extra degrees of freedom, which we will denote as the *parameters* of the transformed equation. The canonical transformation can only be defined up to these parameters. However in the context of a physical model, their values will be constrained by the requirement that the solution of the initial system belong to the Bargmann-Fock space.

The remaining equations in the system (17) i.e. the formulas corresponding to $l = q + 2, q + 3, ...$ form a set of recurrence equations for the $\{a_{rs}(z)\}$ matrix. This system determines $\{a_{rs}(z)\}$ as a function of the parameters of the transformed system. The procedure of determining the transformation matrix for a given set of parameters is in general infinite and it may be very difficult to find the parameters that lead to solutions in the Bargmann-Fock space. Hence it is conceivable that we know the initial (9) and the transformed (13) systems of equations, without being able to solve the transformation matrix (12). In some cases this still allows us to draw some important conclusions about features of physical interest, such as degeneracies or symmetries.

A special case arises if we assume that the expansion (12) of $\{a_{rs}(z)\}$ in negative powers of z is finite. In this case confinement to the Bargmann-Fock space can indeed easily be guaranteed. If the highest order in the denominators of (12) is not greater than ρ_t from (16), which is the lowest power in the expansion of the $F_s^{(t)}(z)$, then the corresponding solution of the initial system is automatically analytic in the origin. In this case the system (17) also remains finite and can be solved. This procedure precisely leads to the isolated exact solutions of the initial system.

IV. The Solution of a Two Level System Coupled to a Single Quantized Mode

A. The canonical form

In this section we derive the canonical form of the dynamical equations for a two level system coupled to a single quantized mode. The Hamiltonian of

such a system, sometimes called Rabi Hamiltonian [4], is of the form

$$H = \omega a^+ a + \mu \sigma_3 + \lambda(\sigma^+ + \sigma^-)(a^+ + a), \tag{19}$$

where a^+ and a are boson field (7) creation and annihilation operators, $\sigma^\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ and σ_1, σ_2, σ_3 are Pauli matrices. The parameter ω is the boson field frequency, 2μ is the atomic level separation and λ is the atom - boson field coupling constant. We choose the energy unit in such a way that $\omega = 1$ and we assume that λ and μ are not vanishing simultaneously.

The first step to solve the system (19) is to make a unitary transformation which replaces operators $\sigma_1 \rightarrow \sigma_3$, $\sigma_3 \rightarrow \sigma_1$ and $\sigma_2 \rightarrow -\sigma_2$. Then we write the stationary Schrödinger equation for the two-component wave function in the position variable $\begin{pmatrix} f_1(\xi) \\ f_2(\xi) \end{pmatrix}$. In the second step, by replacing $a^+ \rightarrow z$ and $a \rightarrow \frac{d}{dz}$, we perform a transition to a Bargmann-Fock space. In this space the Schrödinger equation is equivalent to a system of two first order differential equations for the Bargmann-Fock space functions $f_1(z), f_2(z) \in \mathcal{F}_1$

$$\begin{aligned} \frac{d}{dz}f_1(z) &= \frac{E - \lambda z}{z + \lambda}f_1(z) - \frac{\mu}{z + \lambda}f_2(z) \\ \frac{d}{dz}f_2(z) &= -\frac{\mu}{z - \lambda}f_1(z) + \frac{E + \lambda z}{z - \lambda}f_2(z), \end{aligned} \tag{20}$$

where E is an eigenenergy of $\mathcal{H}$. Note, that in the Schrödinger representation of creation and annihilation operators we have $a^+ \rightarrow \frac{1}{\sqrt{2}}(\xi - ip_\xi)$, $a \rightarrow \frac{1}{\sqrt{2}}(\xi + ip_\xi)$, where ξ and p_ξ are conjugate position and momentum. The system corresponding to (20) in this representation consist of two second order differential equations in a real variable ξ.

The present, Bargmann-Fock space formulation of the problem has been investigated earlier and approximate solutions have been found [4, 9]. In addition Kuś has derived some isolated exact solutions, corresponding to degenerate levels [3].

For our purposes we point out three properties of the system. Firstly, it is of the form (9) with two finite singularities in $z = \lambda, -\lambda$. Secondly, expanding coefficients of (20) in a Laurent series (10) we find that $q = 0$ and

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therefore the singular point at infinity is of the first rank. Finally, we note the following symmetry: if $\begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}$ is a solution of (20) then $\begin{pmatrix} f_2(-z) \\ f_1(-z) \end{pmatrix}$ is also a solution, corresponding to the same energy value.

The Birkhoff theorem is found to apply to the system in (20). Hence one can claim the existence of a canonical form. With $q = 0$, the coefficients of this form will be polynomials of the first degree! They can be obtained from equations (17) and (18), as explained in the previous section. The linear terms of $P_{rs}(z)$ are inferred at once from (18) i.e. $P_{rs}^{(1)}(z) = (-1)^r \lambda \delta_{rs}$. To calculate the remaining four zeroth order terms of $P_{rs}(z)$ we use equation (17) with $l = 1$. The canonical form of the system (20) is therefore

$$z \frac{d}{dz} F_1(z) = \left(E - \lambda z + \lambda^2\right) F_1(z) + \left(-\mu - 2\lambda a_{12}^{(1)}\right) F_2(z)$$

$$z \frac{d}{dz} F_2(z) = \left(-\mu + 2\lambda a_{21}^{(1)}\right) F_1(z) + \left(E + \lambda z + \lambda^2\right) F_2(z), \qquad (21)$$

where $F_1(z)$ and $F_2(z)$ are linearly transformed $f_1(z)$ and $f_2(z)$ (11), and $a_{12}^{(1)}$, $a_{21}^{(1)}$ are parameters belonging to the transformation matrix (12).

The main feature of the canonical system is that it has only one singularity at $z = 0$, and because of that is exactly solvable. Prior to solving it however we exploit the symmetry of solutions $\begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}$ of the initial system (20). To preserve this symmetry in the transformed pair, i.e. if $\begin{pmatrix} F_1(z) \\ F_2(z) \end{pmatrix}$ is a solution of (21) then so is $\begin{pmatrix} F_2(-z) \\ F_1(-z) \end{pmatrix}$, we must impose the following symmetry of the transformation matrix

$$a_{ij}(z) = a_{[i+1][j+1]}(-z), \qquad i, j = 1, 2 \qquad (22)$$

where $[\cdot + \cdot]$ denotes an addition modulo 2.

B. Quantization conditions and canonical solutions

Physical constrains require the solutions of (20) to belong to the Bargmann-Fock space, i.e. to converge in the entire plane. This leads to quantization conditions, as we will show in this section.

The canonical system contains four parameters λ, μ, E and $a_{12}^{(1)}$. In the usual perception of the problem λ and μ are external quantities, describing the physics of the system, while E and $a_{12}^{(1)}$ are essentially free parameters, to be determined by the quantum conditions. The indicial equation (15) corresponding to the system (21) is seen to link E and $a_{12}^{(1)}$:

$$\rho = E + \lambda^2 \pm A, \tag{23}$$

where $A = \mu + 2\lambda a_{12}^{(1)}$. For the system (21) to have solutions in the Bargmann-Fock space at least one value of ρ must be a non-negative integer. The value of A thus determine the whole energy spectrum but in turn it is controlled by the $a_{12}^{(1)}$ parameter, from which the transformation matrix can be generated via (17). For the time being we will treat it as a free parameter, delineating classes of solutions in the energy spectrum. As we will see later its value will be fixed by the requirement that the solutions of the original system (20) are in the Bargmann-Fock space.

The general solution of (21) can be obtained by transforming it into a second order form

$$z^2 F_1''(z) + z\left[1 - 2\left(E + \lambda^2\right)\right] F_1'(z) + \left[\left(E + \lambda^2\right)^2 - A^2 + \lambda z - \lambda^2 z^2\right] F_1(z) = 0. \tag{24}$$

This equation can be reduced to a confluent hypergeometric (or Kummer) equation. Its general solution is a combination of two functions

$$\begin{aligned} F_1(z) &= C_1 \exp(\lambda z) \, {}_1F_1(1 + A, 1 + 2A; -2\lambda z) \, z^{E + \lambda^2 + A} + \\ &\quad C_2 \exp(\lambda z) \, {}_1F_1(1 - A, 1 - 2A; -2\lambda z) \, z^{E + \lambda^2 - A}. \end{aligned} \tag{25}$$

with arbitrary C_1 and C_2. The function ${}_1F_1(a, c; z)$ is called confluent series or Kummer function and is defined for all complex a, z and $c \neq -n$, $n = 0, 1, 2, \ldots$

$$_1F_1(a, c; z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \ldots. \tag{26}$$

The Kummer function is entire. The solutions for $F_2(z)$ are of the same general form (25), with however z replaced by $-z$.

The asymptotic behavior of the solution (25) for $|z| \to \infty$ is restricted by the function $z^{\alpha_1} \exp(\lambda z) + z^{\alpha_2} \exp(-\lambda z)$, where α_1, α_2 are real numbers [10] and therefore the condition (4) is always obeyed. Consequently the solution (25) belongs to the Bargmann-Fock space $\mathcal{F}_1$ provided at least one of the roots of the indicial equation (23) is a non-negative integer.

A particularly simple case occurs if $\lambda = 0$ because then the transformed system (21) coincides with the initial one (20). The transformation matrix reduces in this case to the unit matrix.

Let us now discuss the solutions (25) and their degeneracy for different values of $E + \lambda^2$.

(i) If $E + \lambda^2$ is not an integer nor a half-integer then, to get physical solutions, we should take A such that one of the numbers $\rho_1 = E + \lambda^2 + A$ and $\rho_2 = E + \lambda^2 - A$ is a non-negative integer. In this case, according to the general rule D_1, one of the two functions composing $F_1(z)$ can be entire. The corresponding solution for $F_2(z)$ is also one-dimensional and is given by $F_2(z) = (-1)^{\rho_t + t} F_1(-z)$.

(ii) If $E + \lambda^2$ is a half-integer then we take A also half-integral. Despite the fact that both $E + \lambda^2 + A$ and $E + \lambda^2 - A$ are now integers (rule D_2), The solutions $F_1(z)$ and $F_2(z)$ are still one-dimensional because in this case one of the numbers $1 + 2A$ or $1 - 2A$ is a non-positive integer and the corresponding Kummer function is not defined.

(iii) If $E + \lambda^2$ is an integer we can take the simple choice $A = 0$. Then, for $E + \lambda^2 \geq 0$ each of the solutions is entire and forms a one-dimensional space (rule D_1), but because the system (21) is diagonal we can always take its two linearly independent solutions of the form $\begin{pmatrix} F_1(z) \\ F_2(z) \end{pmatrix}$ and $\begin{pmatrix} F_1(z) \\ -F_2(z) \end{pmatrix}$. Consequently in this case the solutions are degenerate. This class of solutions will comprise the exact solutions found by Kuś.

(iv) If $E + \lambda^2$ is an integer and $E + \lambda^2 > 0$, we can also take A integral, $0 < |A| \leq E + \lambda^2$. In this case, although the system (21) is no more diagonal, solutions $F_1(z)$ and $F_2(z)$ become two-dimensional and lead to degenerate solutions. This represents the rarest case of D_2 when two roots of the indicial equation are integral and both corresponding solutions are analytical.

To summarize we conclude that the only case where solutions are degenerate can occur when $E + \lambda^2$ is a non-negative integer.

C. The isolated exact solutions

In the previous section we have shown that the canonical form of the Rabi Hamiltonian can be solved within Bargmann-Fock conditions. This results in a coupling between A and E parameters, which is interesting in its own right, but does not lead to quantized energies. As we have already mentioned, the true quantization condition stems from the requirement, that the solutions of the original system belong to the Bargmann-Fock space. This implies that the transformation of the canonical solutions must act within the Bargmann-Fock space. In this way we fix A and hence E.

In general this procedure is nontrivial since the transformation matrix is generated by an infinite system of equations (17), yielding an infinite series of coefficients $a_{rs}^{(k)}$. In this section we will not be concerned with the general case but only study the exactly solvable class of solutions, which corresponds to $A = 0$. We assume that the transformation is nontrivial i.e. $\lambda \neq 0$.

Let us first consider the simplest possibility $\mu = 0$. With $A = 0$, and hence $a_{12}^{(1)} = 0$, the indicial equation yields that $\rho = E + \lambda^2$ must be a non-negative integer. It is easy to find that the corresponding transformation matrix is diagonal and reads

$$\{a_{rs}(z)\} = \begin{pmatrix} \left(1 + \frac{\lambda}{z}\right)^{E+\lambda^2} & 0 \\ 0 & \left(1 - \frac{\lambda}{z}\right)^{E+\lambda^2} \end{pmatrix}, \tag{27}$$

whereas the solutions are generated by

$$\begin{aligned} F_1(z) &= \exp(-\lambda z)\, z^{E+\lambda^2} \\ F_2(z) &= \exp(\lambda z)\, z^{E+\lambda^2}. \end{aligned} \tag{28}$$

Note that in this case the expansion (12) in negative powers of both diagonal terms terminate. The last non zero coefficients are $a_{rs}^{(E+\lambda^2)}$. On the other hand the lowest power of z in the expansion of the transformed solution (25)

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is also $E + \lambda^2$. It is then easily verified that the solution of the initial system, which follows from (11), is of form (1) and thus belongs to the Bargmann-Fock space. The energy spectrum is obtained directly from the indicial equation (23) and reads $E_\rho = \rho - \lambda^2$, where $\rho = 0, 1, 2, \dots$.

In the line of this example we now turn to the more general case $\mu \neq 0$ but keeping the $a_{rs}^{(k)}$ matrix finite. As a first example we assume that the expansion (12) of the transformation matrix terminates, and $a_{rs}^{(1)}$ are the only non-zero coefficients. In other words $a_{rs}^{(2)} = a_{rs}^{(3)} = \dots = 0$. In this case the system (17) is also finite because starting from $l = 3$ all the higher order equations are equivalent to equations corresponding to $l = 2$. Taking into account symmetry properties (22) of the matrix, the system reduces to four equations

$$
\begin{aligned}
\bar{a}_{11} + 2\mu\bar{a}_{12} + 2\lambda\bar{a}_{12}^2 - \lambda(E + \lambda^2) &= 0 \\
2\mu\bar{a}_{11} + \bar{a}_{12} + 2\lambda\bar{a}_{11}\bar{a}_{12} + \lambda\mu &= 0 \\
(E + \lambda^2)\bar{a}_{11} + \mu\bar{a}_{12} - \lambda(E + \lambda^2) &= 0 \\
\mu\bar{a}_{11} + (E + \lambda^2)\bar{a}_{12} + \lambda\mu &= 0
\end{aligned}
\tag{29}
$$

where we denote $\bar{a}_{kl} = a_{kl}^{(1)}$ for convenience. Let μ be non trivial and $0 < \mu \leq (E + \lambda^2)$. Then, after some algebra, one can show that the system (29) is equivalent to the transformation matrix

$$
\{a_{rs}(z)\} = \begin{pmatrix} 1 + \frac{1 + \mu^2}{4\lambda z} & -\frac{\mu}{2\lambda z} \\ \frac{\mu}{2\lambda z} & 1 - \frac{1 + \mu^2}{4\lambda z} \end{pmatrix}
\tag{30}
$$

and two other conditions: $E + \lambda^2 = 1$ and $4\lambda^2 + \mu^2 = 1$. From the $a_{12}(z)$ element in (30) it also follows that $A = \mu + 2\lambda a_{12}^{(1)} = 0$.

Thus the very assumption that the transformation matrix terminates gave us the form of the matrix, the values of $E + \lambda^2$ and A, showing that they correspond to a degenerate solution of (25). In addition the system has energy $E = 1 - \lambda^2$ only if the atomic level separation 2μ and the atom-boson field coupling λ obey condition $4\lambda^2 + \mu^2 = 1$. The corresponding eigenfunction $\begin{pmatrix} F_1(z) \\ F_2(z) \end{pmatrix}$ is given by (28) and $\begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}$ can be found by applying (11). One

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can easily shown that $f_1(z)$ and $f_2(z)$ belong to the Bargmann-Fock space. This solution agrees with the first root of the Kuś series [3].

In an analogous way one can show that if the transformation matrix terminates from the third order coefficients onwards $a_{rs}^{(3)} = a_{rs}^{(4)} = ... = 0$ and $0 < \mu \le (E + \lambda^2)$ then the transformation matrix functions are

$$
\begin{aligned}
a_{11}(z) &= a_{22}(-z) = 1 + \frac{2\lambda + \frac{\mu^2}{2\lambda}}{z} + \frac{\lambda^2 + \mu^2 + \frac{\mu^4 - \mu^2}{8\lambda^2}}{z^2} \\
a_{12}(z) &= a_{21}(-z) = -\frac{\mu}{2\lambda z} + \frac{\mu(6\lambda^2 + \mu^2 - 1)}{4\lambda^2 z^2}
\end{aligned}
\tag{31}
$$

i.e. again $A = 0$ and the remaining conditions are $E + \lambda^2 = 2$ and $32\lambda^4 - 32\lambda^2 + 12\lambda^2\mu^2 - 5\mu^2 + \mu^4 + 4 = 0$, which corresponds to the second root of the Kuś series. In this way all solutions of the Rabi Hamiltonian characterized by a terminating $\{a_{rs}(z)\}$ can be generated. They are found to coincide with all known exact solutions.

V. Discussion

The Rabi Hamiltonian describes the coupling between a two level system and a single mode through a linear interaction term $\lambda\sigma_1(a^+ + a)$. This problem is relevant in quantum optics but also appears in molecular physics as the simplest example of vibronic coupling [11, 12].

In Bargmann-Fock space it can be represented by a system of two first order differential equations, as shown in (20). For $\mu = 0$ this system separates into two independent equations

$$
\begin{aligned}
(z + \lambda)\frac{d}{dz}f_1(z) &= (E - \lambda z)f_1(z) \\
(z - \lambda)\frac{d}{dz}f_2(z) &= (E + \lambda z)f_2(z).
\end{aligned}
\tag{32}
$$

This situation corresponds to a highly symmetric case with two uncoupled harmonic oscillators that are displaced to the left and to the right in coordinate space [9], and cross at the coordinate origin. From this perspective

the introduction of the level separation 2μ may be viewed as a symmetry lowering perturbation, which couples the two oscillators.

If we now compare the original system (20) to the canonical one (21), we still have essentially a set of two displaced oscillators - but now the coupling term no longer corresponds to μ but to $\mu + 2\lambda a_{12}^{(1)}$! Hence the canonical transformation provides a degree of freedom that allows to perform a displacement in the space of the coupling parameter itself. We can thus adjust $a_{12}^{(1)}$ in such a way that it compensates for the energy gap, by requiring $a_{12}^{(1)} = \frac{-\mu}{2\lambda}$, or $A = 0$. System (21) then becomes

$$
\begin{aligned}
z\frac{d}{dz}F_1(z) &= (E - \lambda z + \lambda^2)F_1(z) \\
z\frac{d}{dz}F_2(z) &= (E + \lambda z + \lambda^2)F_2(z).
\end{aligned}
\tag{33}
$$

This system again describes two degenerate harmonic oscillators that are displaced in coordinate space and also underwent a translation in z space over $+$ or $-\lambda$.

We thus have shown that the exact solutions , that are found for $A = 0$, can be mapped onto the energy spectrum of a degenerate harmonic oscillator. A similar result can also be obtained for the Juddian exact solutions of the $E \otimes e$ Jahn-Teller Hamiltonian [13].

Acknowledgments

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